ON USING THE LOG-HYPERBOLIC DISTRIBUTION TO DESCRIBE THE TEXTURAL CHARACTERISTICS OF EOLIAN SEDIMENTS[1]

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ABSTRACT: Following the early work of Bagnold (1941) it has recently been suggested that the log-hyperbolic distribution offers a better description of grain-size distribution than the log-normal distribution. This paper undertakes a comparison of the suitability of the log-hyperbolic and log-normal distributions in describing the textural characteristics of desert-dune sediments. A statistical analysis of the results shows that no apparent gain is obtained when using the parameters of the log-hyperbolic distribution to describe such sediments.

INTRODUCTION

The ambiguities and difficulties involved in using the grain-size distribution characteristics of a sediment, in order to relate that sediment to a specific depositional environment, are deeply imbedded in the sedimentological literature (e.g., Sedimentation Seminar 1981). The empirical nature of much of this work, and the adoption, at times, of a battery of summary statistics, which bear a distant relationship to the sediment-transport processes involved, generates skepticism in the results obtained (see, for example, the discussion by Folk 1977 and Picard 1977 of the paper by Freeman and Visher 1975). Consequently, it is not surprising that, for some researchers, particle-size analysis is now seen as a largely futile exercise (Ehrlich 1983). But while there are grounds for criticism, the empirical and inductive nature of comparative sediment grain-size work is understandable. The incomplete understanding of sediment transport mechanics, and the difficulties imposed by sediment availability and supply, have hindered attempts to develop a more deductive and physically based study of grain-size distributions. Notwithstanding the criticism directed at the use of grain-size distribution characteristics to describe and/or characterize specific depositional environments, the need will continue for such work to be used as one indicator of depositional history of a sediment. This requires that a grain-size distribution is adequately described by a suitable model distribution. Furthermore, it is to be expected that the model distribution stimulates the development of a physical explanation for the sediment grain-size distribution. The recent work of Barndorff-Nielsen (1977), Bagnold and Barndorff-Nielsen (1980), Barndorff-Nielsen and others (1982), and Barndorff-Nielsen and others (1983) is welcomed on both counts. It is based on earlier work by Bagnold (1941) and claims that the log-hyperbolic distribution offers a better description of sediment grain-size distributions than the customarily used, but apparently often poorly fitted, log-normal distribution. This claim suggests that the log-hyperbolic distribution will provide summary statistics which more adequately relate a sediment to a specific depositional environment.

So far there have not been any real attempts at using the log-hyperbolic distribution in a routine sedimentological context. Our aim in this paper is to examine the applicability of the log-hyperbolic distribution to "type" the textural characteristics of longitudinal desert-dune sediments, and to evaluate the ability of this distribution to distinguish between the three dune settings—dune sides, dune crests, and interdunal corridor. The results obtained from the log-hyperbolic distribution are compared with those obtained from the conventional log-normal distribution. Both distributions are fitted to the data by the method of maximum likeness proposed by Barndorff-Nielsen (1977).

SEDIMENT SAMPLING AND PREPARATION

The presently stable and vegetated longitudinal dunes of the Gascoyne River–Exmouth Gulf area of northwestern Australia were sampled along traverses, with sediments obtained from the middle section of the dune sides, the dune crest, and the interdunal corridor. Five traverses in different parts of the dune field were undertaken. The samples were obtained from approximately 0.5 m below the surface, and a grab sample of about 200–500 g was usually collected.

When establishing the grain-size distribution of sediments through sieving, the sample weight adopted must clearly be a function of the grain-size variation in the sample. Because in this form of analysis, weight-frequency is taken as a surrogate for particle-frequency, the statistical problems associated with determining the necessary sample size (weight) are formidable. Consequently,

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only a "rule of thumb" approach is generally taken to determine sample size (e.g., Blatt et al. 1972, p. 47; Folk 1974, p. 33). In this study of the textural characteristics of longitudinal dunes, Folk (1971) collected a one-inch-square cube of material (~ 15–20 g). He adopted this sample size because he considered that this (1) is not such an excessive weight as to overload the screens; and (2) this approximates the cross-sectional area of an average thin section, facilitating comparison with the size parameters of lithified eolian sandstone. Tsoar (1978), using micro-sieves, used sample amounts of as little as 5 to 10 g. More recently, Binda (1983) used sample weights of 100 g and 60–70 g.

During the preparation of the obtained samples for sieving, and also during preliminary sieving, it became obvious that the longitudinal dune sands contained small amounts of material which were larger than $-0.5\ \phi$. This sized material formed such a small percentage of the total sample that the likelihood of obtaining some indication of this in a sample of 20 g (the original sample weight adopted) was thought to be remote. For this reason it was considered necessary to adopt a larger sample weight. Van Rooyen and Burger (1974) suggest a 100-g sand sample sieved at $0.5\ \phi$ intervals for 30 minutes. This procedure was adopted for the present study with a lower screen size of $4.5\ \phi$, but reservations existed about whether a large sample weight would not lead to screen overloading; with a weight retained by individual screens exceeding 30 g in some samples, this was seen as a potentially serious problem. To overcome this, after mechanical sieving, each screen was further hand-sieved and carefully "worked" with a fine brush to make sure that no grains smaller than the screen size were retained. This procedure was tedious and extremely time consuming, but it was felt that this was the only way such a large sample weight could be processed so that it was possible to be confident of the validity of the results.

After sieving, a sample from each screen was inspected under a binocular microscope to make sure that aggregates did not occur in the sample. In a number of samples, carbonate cementation of grains had occurred, and in these cases the sample was disaggregated by standard techniques (Folk 1974, p. 17), with the sample subsequently washed in distilled water. After such treatment, the screen sample was again viewed under a binocular microscope to check for grain aggregation.

THE NORMAL AND HYPERBOLIC DISTRIBUTIONS

The well-known normal distribution has two parameters and a probability density function which may be written

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}.$$

It is characterized by the fact that a plot of log p against x gives a parabolic curve. The parameters μ and σ give location and scale, respectively.

The hyperbolic distribution was introduced formally by Barndorff-Nielsen (1977) and has been further discussed and summarized by Bagnold and Barndorff-Nielsen (1980) and by Barndorff-Nielsen and others (1982). It has four parameters and the probability density function

$$p(x;\mu,\delta,\phi,\gamma) = \frac{\sqrt{\phi\gamma}}{\delta(\phi+\gamma)K_1(\delta\sqrt{\phi\gamma})} \cdot\exp\left\{-\frac{\delta}{2}\left[(\phi+\gamma)\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2} - (\phi-\gamma)\frac{x-\mu}{\delta}\right]\right\}.$$

Here K_1 is a modified Bessel function. The hyperbolic distribution is characterized by the fact that plotting log p against x gives a (possibly asymmetric) hyperbola.

Like the parameters of the normal distribution, μ and δ give location and scale. All four hyperbolic parameters may be given geometric interpretations in terms of the hyperbolic graph. The parameters ϕ and γ give the slopes of the left and right-hand linear asymptotes of log p, and μ is the abscissa of the intersection point of the two asymptotes. The parameter δ is related to the graph through the derived parameter

$$\zeta = \delta\sqrt{\phi\gamma},$$

which is the difference between the ordinate of the intersection point of the two asymptotes and the ordinate of the hyperbolic curve at the mode point.

The grain-size distributions, both the normal and the hyperbolic, are applied to the log-grain size. Hence we refer to the log-normal or log-hyperbolic distribution for grain size. In this study we apply these theoretical distributions to the mass-size sample distributions rather than to the frequency-size distribution. It is theoretically possible for both mass-size and frequency-size sample distributions to follow log-hyperbolic distributions (with different asymptotic slopes), but not for both to be log-normal.

When comparing the normal and hyperbolic distributions, it will be important to note that the normal is a limiting case of the hyperbolic. The curvature of the hyperbolic curve at its mode point is given by

$$\tau^2 = \frac{(\phi\gamma)^{3/2}}{\delta\left(\frac{\phi+\gamma}{2}\right)^2}.$$

By making ϕ, γ, and δ large in such a way that τ^2 and $\phi - \gamma$ remain fixed, the resulting hyperbolic distribution can be made arbitrarily close to the normal distribution with mean $\mu + (\phi - \gamma)/(2\tau^2)$ and variance $1/\tau^2$. Any normal distribution may be approximated in this way.

FITTING DISTRIBUTIONS TO MASS-SIZE DATA

The mass-size distribution of a sand sample can be displayed as a double-logarithmic histogram plot as described in Barndorff-Nielsen (1977). To fit a log-hyperbolic distribution to the data we could simply plot a hyperbolic curve through this histogram and estimate the parameters directly from the curve on the basis of their geometric interpretations. A similar process would be to obtain the graphic measures of a log-normal distribution as in Folk and Ward (1957). Alternatively, as is commonly done, we could fit the log-normal distribution by equating the first and second moments of the fitted and sample distributions. Neither of these methods is as sensitive as maximizing (or minimizing) an appropriate statistical objective function.

If the data consisted of counts of independent observations, then the principle of maximum likelihood would lead us to maximize

$$L = \sum n_i \log p_i ,$$

where p_i is the theoretical probability mass and n_i is the number of observations in the ith interval. When fitting to mass-size distributions of dune sediments, likelihood methods are not available to us, partly because our data consist of weights rather than counts, but more importantly because the sizes of the individual sand grains cannot be considered independent. Following Barndorff-Nielsen (1977), we maximize L even though it is not a likelihood, simply substituting relative weights r_i for the counts n_i. We justify the method instead in terms of information theory. Maximizing L, with weights substituted for counts, is equivalent to minimizing

$$I = \sum r_i \log \frac{r_i}{p_i} .$$

Kullback (1959) calls I the mean information for discrimination in favor of the r_i against the p_i. It may be interpreted as a measure of dissimilarity or distance between the theoretical and empirical probability masses. In this paper we actually use

$$D = 2W \sum r_i \log \frac{r_i}{p_i} ,$$

where W is the total weight of the sample, which as an objective function is also equivalent to L. We call the minimum of D the divergence of the fitted distribution from the data.

It should be added that fitting the hyperbolic distribution to data was found to be an extremely ill conditioned numerical problem, and considerable programming effort was expended to get reliable estimates for the samples.

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00/903556789
02/01233444555667700112233345668888
04/013678889124777
06/0001347711256
08/8449
10/83
12/3
14/
16/
18/
20/17
22/3958
24/
26
28/0
```

A) ALL NORMAL

```
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02/1111235667789000166799
04/00012222368891278
06/01135137
08/1124045
10/67
12/00
14/
16/0
18/0
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B) ALL HYPERBOLIC

```
00/39
02/04560125688
04/064
06/03
08/8
10/3
12/3
14/
16/
18/
20/
22/8
24/
26/
28/0
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C) CREST NORMAL

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00/7145557789
02/3580167
04/0012
06/
08/
10/
12/00
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D) CREST HYPERBOLIC

```
00/905567
02/3445601233468
04/137977
06/7711
08/
10/
12/
14/
16/
18/
20/
22/3
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E) SIDE NORMAL

```
00/7888902369
02/127900
04/2238127
06/01317
08/2
10/
12/
14/
16/0
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F) SIDE HYPERBOLIC

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00/9
02/12738
04/827
06/01425
08/49
08/49
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G) CORRIDOR NORMAL

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00/47
02/11676
04/2698
06/3
08/104
10/7
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H) CORRIDOR HYPERBOLIC

FIG. 1.—Stem-leaf plots of the mean divergences. The unit and first decimal place are used as the "stem," the second decimal place as a "leaf."

Computer Routines Used

Programming was carried out in FORTRAN on a DEC KL10 computer using library routines from a variety of sources.

For the normal distribution the divergence was minimized by the EM algorithm for maximum likelihood estimation with incomplete or grouped data described by Dempster, Laird, and Rubin (1980). The NAG (Numerical Algorithms Group 1983) routine S15ABF was used for the normal densities.

For the hyperbolic distribution, minimization used a quasi-Newton algorithm described by Fletcher (1970) and

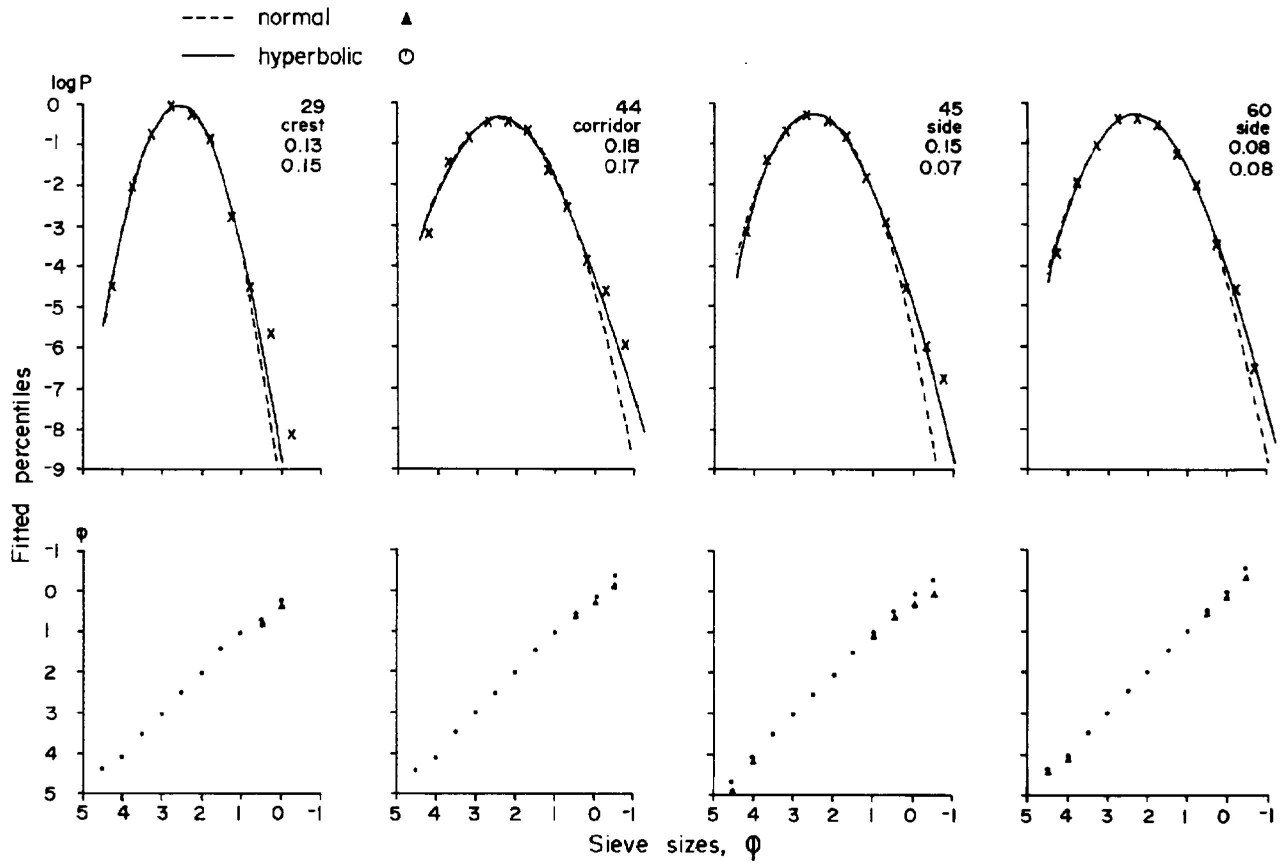

FIG. 2.—Grain-size distributions of four samples to which both the normal and hyperbolic distributions fit well. The sample and fitted curves are displayed above as a double-log histogram, below as a Q-Q plot. Also given are the normal (above) and hyperbolic (below) mean divergences.

programmed as subroutine VA09A for the Harwell subroutine library (Hopper 1973). Starting values were obtained by assuming the sample mass to be concentrated at the midpoints of the sieve intervals.

The theoretical probability masses P_i, and their partial derivatives, were computed using the NAG numerical integration program D01AHF. Functions of the modified Bessell functions appearing in the hyperbolic density and its derivatives, were programamed from the polynomial approximations of Abramowitz and Stegun (1970), and from the NAG routines S18ACF and S18ADF. Percentiles of the hyperbolic distribution were obtained by the Newton method for solving nonlinear equations, and those of the normal distribution from the IMSL (IMSL 1982) routine MDNRIS.

COMPARING THE NORMAL AND HYPERBOLIC DISTRIBUTIONS

As described above, the hyperbolic distribution has two more parameters than the normal, and includes the normal as a limiting case. For this reason the fitted hyperbolic distribution should appear to match any given sample at least as well as the best normal. In fitting the extra parameters, however, we incur costs which include increased statistical variability of the estimates, increased correlation between those estimates, and a considerable increase in programming complexity. The hyperbolic distribution, moreover, appears to be more erratic in the presence of error or of data which actually follows neither distribution. The upper and lower asymptotes and γ in particular are very sensitive to small changes in the upper and lower tails of the sample (see the remarks made about distributional tails in the discussion). We must judge whether the extra parameters produce an improvement in fit which is worth the cost; that is, whether the hyperbolic distribution is a model of a departure from the normal which is not purely random.

Since the sizes of individual grains are not independent, and because the nature of their dependence is not known, we cannot calculate the sampling variability of the observed mass-size distribution. For this reason, neither we nor Barndorff-Nielsen and his coworkers offer formal significance tests of goodness of fit. (Barndorff-Nielsen et al. 1982 have succeeded, though, in providing standard errors for the hyperbolic parameters through the artifact of duplicate samples.) Instead we must make judgements subjectively from visual displays of each sample.

Nevertheless we attempt to order the samples in terms of goodness of fit on the basis of their mean divergences. The divergence D defined above is the statistic which would be chi-square distributed if the data were inde-

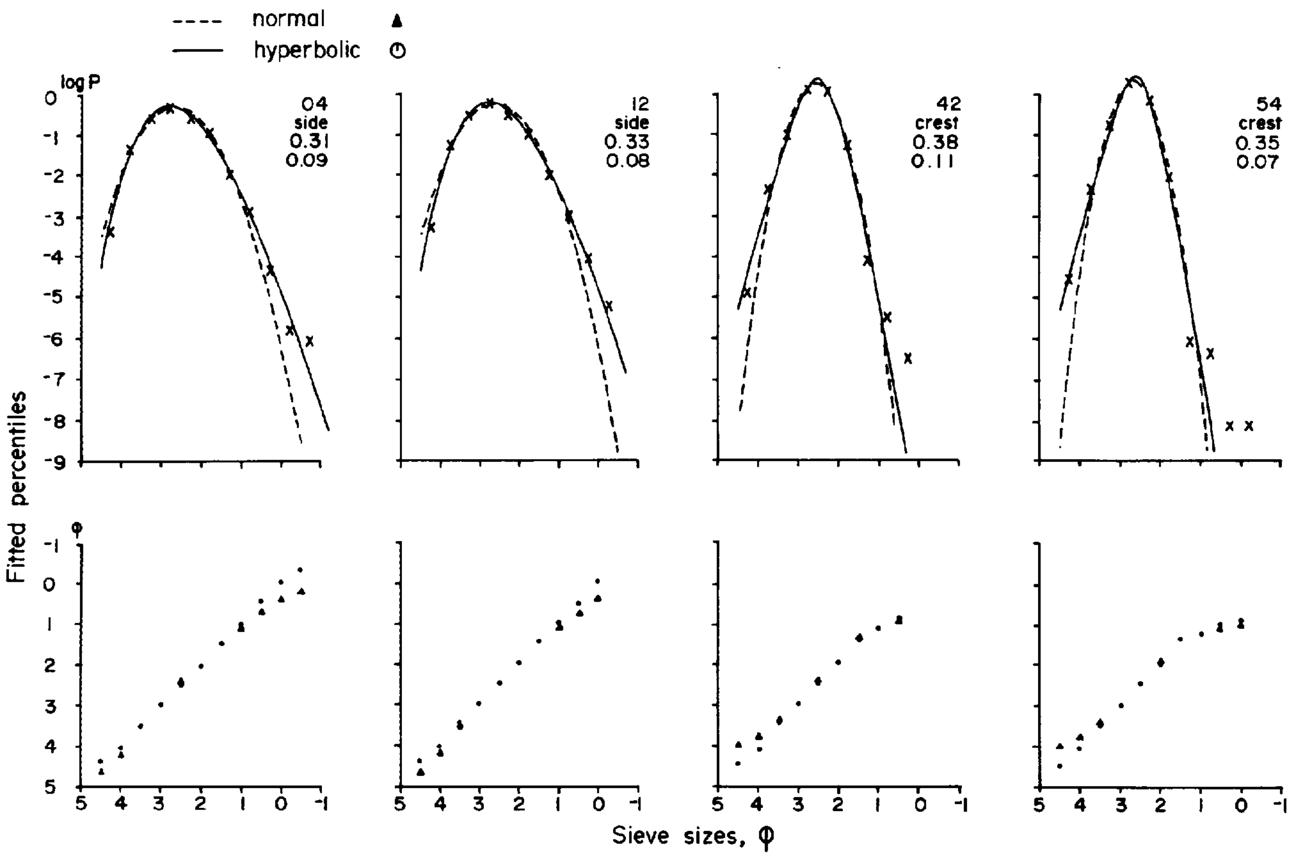

FIG. 3.—Grain-size distributions of four samples to which the hyperbolic distribution fits rather better than the normal. The sample and fitted curves are displayed above as a double-log histogram, below as a Q-Q plot. Also given are the normal (above) and hyperbolic (below) mean divergences.

pendent counts instead of relative weights. To correct for the number of parameters estimated, we define the mean divergence M to be the divergence divided by the number of degrees of freedom. The number of degrees of freedom is the number of nonzero frequencies less the number of estimated parameters, two for the normal distribution, four for the hyperbolic. This definition may be motivated by the observation that a normal curve may be chosen to fit virtually perfectly any sample with only two nonzero frequencies, while a hyperbolic may be chosen to fit perfectly any unimodal sample with only four nonzero frequencies.

RESULTS

A total of 83 samples was included in the study. Of these, 23 were dune crest samples, 30 were dune side, and 16 were interdunal corridor. The resulting mean divergences ranged from 0.07 to 2.80.

Stem-leaf plots of the mean divergences are given in Figure 1. These plots are horizontal histograms with the unit and first decimal place (rounded to an even integer) as "stems" and the second decimal place as "leaves." Although the hyperbolic distribution produces more very small divergences (six below 0.10 instead of one) and fewer very large (none above 1.80), on the whole the spread of the mean divergences for the two distributions is similar. Both distributions fit the dune-crest and dune-side samples better than they do the dune corridor, and the hyperbolic fits the dune-crest samples best of all.

Figures 2 to 5 give plots for a selection of the samples. For each sample displayed we give Q-Q plots as well as the double logarithmic histogram plots. The Q-Q plots compare the sample and fitted cumulative distributions by plotting the sieve sizes against the fitted percentiles for the proportion weight retained by the sieve. This is similar to plotting the cumulative sample distribution on normal or hyperbolic probability paper. Note that it is not possible to represent the pan weight or the weight passing through the finest sieve on the histogram, but both are evident on the Q-Q plot.

Figure 2 displays plots of four samples which both distributions fit well. Figure 3 plots four samples which are fitted rather better by the hyperbolic distribution. Figure 4 plots two samples which both distributions fit poorly. Figure 5 plots two samples which are fitted well except for a tendency towards bimodality. The closeness of the fitted normal and hyperbolic curves depends on the size of the hyperbolic scale parameter δ. Whenever it is large, the fitted curves are almost indistinguishable.

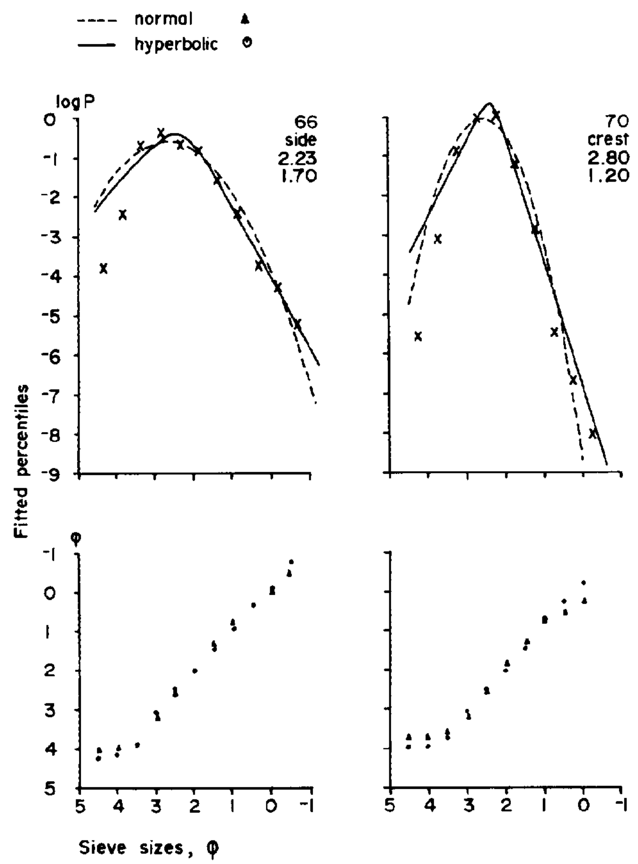

FIG. 4.—Grain-size distributions of two samples to which both the normal and hyperbolic distributions fit poorly. The sample and fitted curves are displayed above as a double-log histogram, below as a Q-Q plot. Also given are the normal (above) and hyperbolic (below) mean divergences.

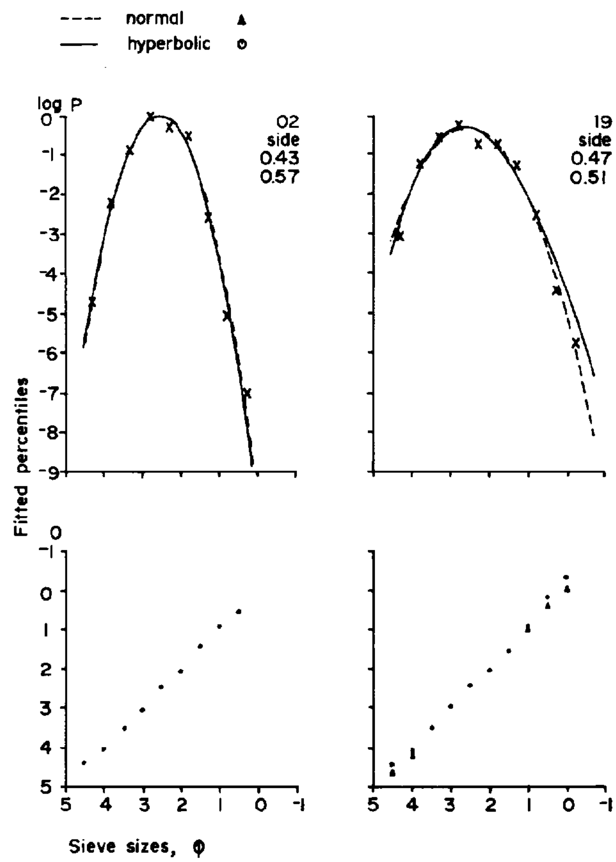

FIG. 5.—Grain-size distributions of two samples which show a tendency towards bimodality, but to which both the normal and hyperbolic distributions otherwise fit well. The sample and fitted curves are displayed above as a double-log histogram, below as a Q-Q plot. Also given are the normal (above) and hyperbolic (below) mean divergences.

The fact that the hyperbolic distribution often fits the dune crest and dune side samples very well is consistent with the Bagnold (1941) and Barndorff-Nielsen (1977) assertion that it approximates well the grain-size distribution of "regular sands." But it is also true that the log-normal distribution shows a high degree of accord with many of the samples. This is sufficient to reject the view of Bagnold and Barndorff-Nielsen (1980) that frequencies obtained from a real sample are unlikely ever to plot as a straight line on normal probability paper.

About one-third of the samples were either bimodal or showed the same sort of irregularity just to the right of the mode that Samples 2 and 19 do in Figure 5. Inability to make a model of bimodality is clearly a shortcoming of both distributions.

Another source of poor fit was large pan weights. This seriously affected eight samples, including Samples 66 and 70 in Figure 4.

Of the two normal parameters, the standard deviation σ best separated the three dune environments, while of the hyperbolic parameters the scale parameter δ did best. In order to incorporate all the parameters of each distribution, the canonical variates for discrimination were calculated. Figures 6 and 7 give scatter plots, together with approximate 95% confidence circles about the environment means. The theory of canonical variates is explained, for example, in Sections 11.5 and 12.5 of Mardia and others (1979). Their effect is to project the parameter vectors onto the plane which contains the three environment means, and to standardize so that each point has unit variance. Implicit in this is the assumption that the parameter vectors have multinormal distributions with a common covariance matrix. To improve the approximation to these assumptions, logarithmic transformations were made of the hyperbolic slope and scale parameters δ, ϕ, and γ, and of the normal standard deviation σ.

The Wilk lambda statistic, calculated from the canonical correlations, showed that both the parameters of both distributions separated the environments at very high levels of significance, higher for the normal. Figures 6 and 7 show that both distributions clearly distinguish the dune-crest samples, but that the dune-side and dune-corridor samples overlap. Comparing Figure 6 with Figure 7 shows that all three environments are slightly better separated by the normal parameters.

These qualitative results are confirmed by the number of samples correctly classified. Maximum-likelihood dis-

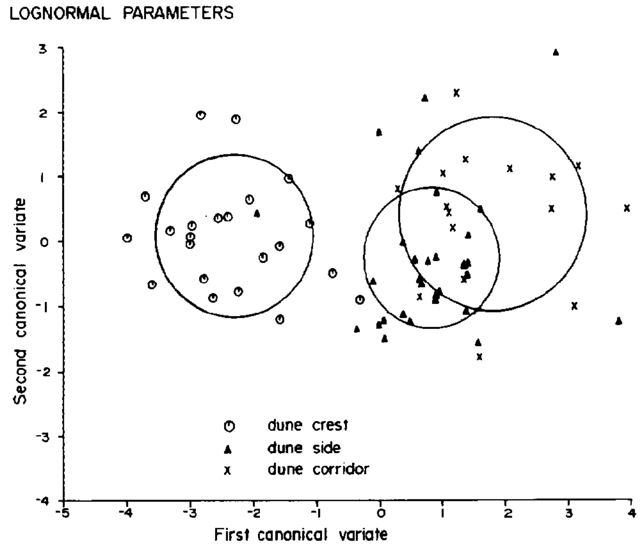

FIG. 6.—Scatter plot of the canonical variates for discrimination, calculated from the estimated normal parameters, together with 95% confidence circles for the three environment means.

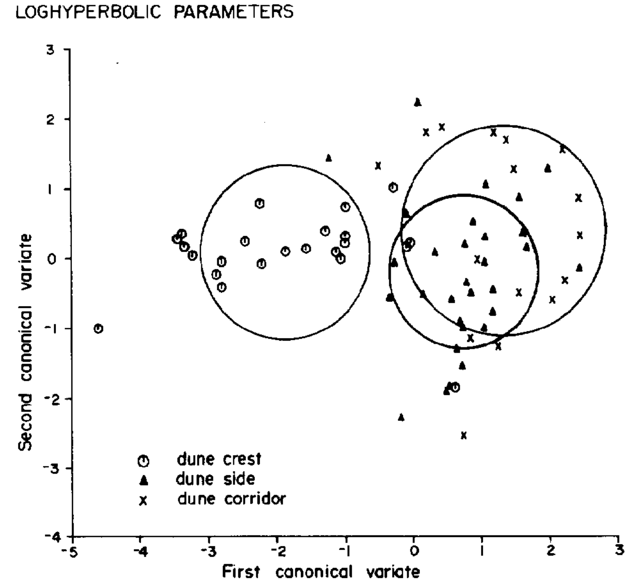

FIG. 7.—Scatter plot of the canonical variates for discrimination, calculated from the estimated hyperbolic parameters, together with 95% confidence circles for the three environment means.

criminant analysis classified each point on the canonical variate scatterplot to the environment, the mean of which it is closest to. The normal parameters correctly classify 21 of the 23 dune-crest samples, 22 of the 30 dune-side samples and 11 of the 16 dune-corridor samples. The hyperbolic parameters correctly classify 19 of the crest samples, 20 of the side samples and 10 of the corridor samples. These counts in fact overestimate the proportion of new samples which would be correctly classified using the same discriminant rule. If allowance were made for the overestimation, the difference between the normal and hyperbolic distributions would be increased.

We believe that the better discrimination obtained from the normal parameters is due to better approximation to the normality and equal covariance assumptions, rather than to greater fidelity to the true mass-size distribution. This highlights the fact that fitting the hyperbolic distribution is a delicate procedure that makes great demands on the data. In any case, no advantage is had in this instance from the hyperbolic parameters over the normal.

DISCUSSION

From the above results it is concluded that no clear gain is obtained by characterizing the grain-size distributions of eolian sediments by the adoption of the log-hyperbolic distribution. The results indicate that the criticisms directed at the suitability of the log-normal distribution in describing sediment grain-size distributions are equally applicable to the log-hyperbolic distribution.

An important reason for the adoption of the log-hyperbolic distribution is the poor accord that the extremes of sediment grain-sizes show with the log-normal model (Bagnold and Barndorff-Nielsen 1980). It is claimed that the log-hyperbolic model encompasses these extreme values much better. In this context the degree of resolution that sieving allows has to be borne in mind. It is recognized that sieving sorts the sediment particles by shape as well as by size (Rittenhouse 1943; Moss 1972; and Komar and Cui 1984). Moss has pointed out that individually weighed suites of quartz particles with very small size (volume) ranges were found to rest on four successive quarter-phi sieves. They were found to be fractionated by the sieves according to shape, with highly flattened particles coming to rest up to four sieves above highly elongated ones of the same size. Hence, sieve analysis must show purely technological coarse and fine "tails," the former due to flattened particles and the latter to elongated particles. It is therefore apparent that extremes of a grain-size distribution are as much an artifact of the technique, as they may be physical entities resulting from the mechanics of sediment transport.

In a tone of censure, Bagnold and Barndorff-Nielsen pointed out that Bagnold drew attention to the log-hyperbolic distribution in 1937—"... but so strong is the inertia of tradition that the implications aroused but little interest until recently" (Bagnold and Barndorff-Nielsen 1980, p. 200). While there would be little point in retaining the log-normal simply because of convention, our results indicate that little is to be gained by adopting the log-hyperbolic distribution. The claims that have been made for the suitability of the log-hyperbolic distribution for describing the grain-size distributions of eolian sediments, and for the routine characterization of the textural characteristics of depositional environments, require qualification.

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